Creating 3D virtual models of buildings

Key Stage: 3

Strand: Measures, Shape and Space

Learning Unit: Trigonometry
Inquiry and investigation

Objectives:
(i) To enrich students' experience in applying trigonometric ratios in real-world scenarios
(ii) To enhance students' abilities in identifying and making assumptions in modelling
(iii) To create a virtual 3D model of a building using mathematics software

Prerequisite Knowledge: Understanding the use of trigonometric ratios in solving problems related to plane figures

Resources Required:
(i) Tape measures and digital protractors
(ii) Desktop or tablet computers with GeoGebra software or Internet connection

Suggestion on Cross KLA Collaboration:
Digital protractors in the form of mobile apps will be used in this series of activities. Alternatively, teachers can consider collaborating with Technology Education (TE) teachers whose lessons can guide students through creating their own digital protractors using, for example, App Inventor or micro:bit.

Background Information:
The estimation of building height is a significant application of mathematics in the field of architecture, engineering, and urban planning. From construction projects to urban development planning, accurate height estimation is essential. The main goal of the following modelling activities is to enrich students' understanding and practical application of the concept of similar 2D figures and trigonometric ratios in problem-solving.

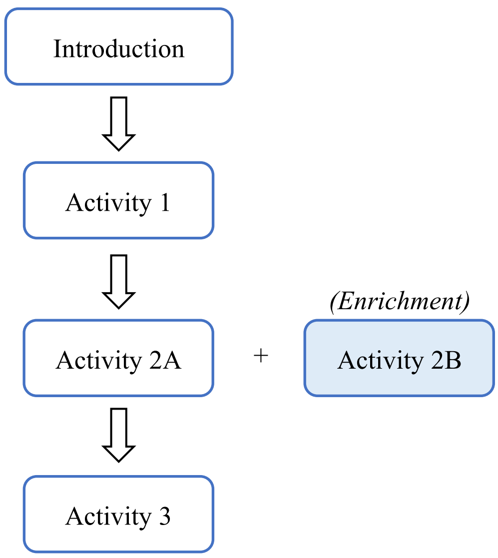
Rooted in descriptive modelling, this set of activities guides students to collect data through measurement and create virtual 3D models of real-world objects. With the aid of information technology, virtual simulations of measurements are provided. Thus, these activities can also be conducted within classroom settings, irrespective of weather conditions. Throughout the series of activities, students not only experience the process of estimating the height of buildings but also engage in discussions that explore facets from assumptions to limitations and situational constraints of estimation and modelling approaches.

Description of the Activities:

There are three main activities in this resource package:

- Activity 1: To discuss a mathematical model of height estimation.
- Activity 2: To estimate heights using trigonometric ratios, when measuring horizontal distances is feasible (2A) or not entirely feasible (2B).
- Activity 3: To create a virtual 3D model of a building.

Based on student abilities and school contexts, teachers can consider adopting the following approach to activity plan.



Based on Yong et al.'s (2015) framework of the mathematical modelling process, the following table summarises the elements that teachers can discuss with students in the corresponding questions.

Phase	**Element**	**WS1**	**WS2A**	**WS2B**	**WS3**
Define	Define the question of interest	Cover page			
	Identify variables and parameters	2	3(b)	7(b)	
Translate	Identify governing principles	2	1, 2, 3(b)	6, 7(b)	
	Make simplifying assumptions	1(b), 3	3(a)	7(a)	4
	Formulate mathematical model	2			3
Analyse	Select appropriate math tools & Solve mathematical problem	1(a), 2, 4	1–4	5–7	1, 2
	Determine or estimate parameters	2	3(b)	7(b)	
	Validate solution				
Interpret	Visualise solution		3	7	3
	Draw appropriate conclusions & Communicate results	2	3	7	3

Activity 1 (refer to Worksheet 1)

The aim of this activity is to establish the context of height estimation through discussing a possible mathematical model.

Pedagogical recommendations:

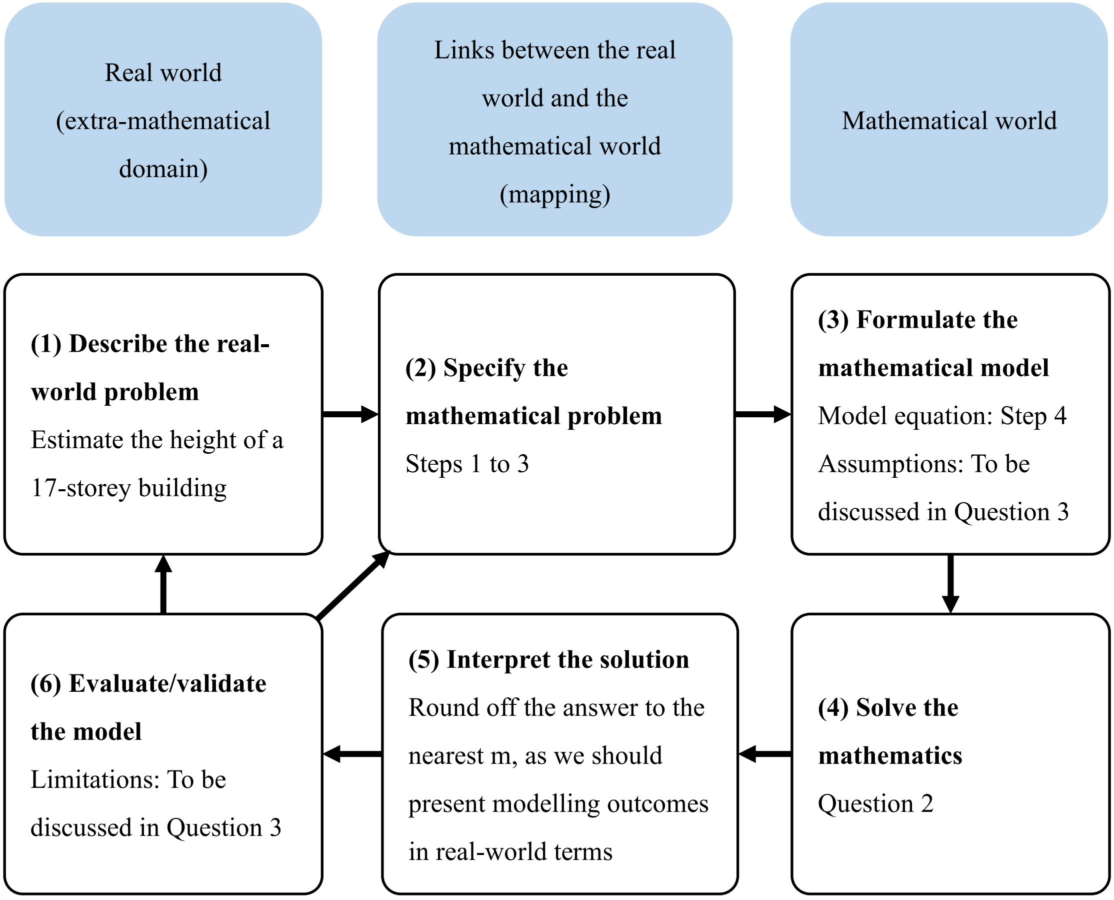
1. The teacher can arouse students' interest by discussing the ways of estimating the length (Question 1(a)) and the height (Question 1(b)) of a podium. This learning experience will be useful to prepare students for the subsequent activities.

   Question 1(a) provides information of equal length in each step. Therefore, the overall length of the podium can be found by multiplication.
   Suggested solution:
   GH
   $= 60 \text{ cm} \times 3$
   $= 180 \text{ cm}$

   Question 1(b) presents an estimation by multiplying the height of one step by the total number of steps. Unlike Question 1(a), equal height in each step is not provided. Therefore, students are tasked to identify this main assumption in the height estimation and discuss in groups whether this assumption is reasonable. The following are some possible discussion outcomes.

   - The student assumes that each step is of equal height (32 cm).
     The assumption is reasonable because of design consistency. In many structures, stairs are often designed to have equal step heights. This design consistency helps ensure safety and ease of use, as people tend to expect uniformity in step heights when climbing stairs.
     [Students may argue that the assumption is not reasonable. For example:
     The assumption is not reasonable because of possible variations in real-world scenarios. Variations can occur due to factors such as customised design, construction errors, or wear and tear over time.]

2. Question 2 presents an approach to estimating the height of a building. The teacher can introduce students to the approach using the mathematical modelling framework (Galbraith & Holton, 2018) as shown in the following figure.



This question focuses on solving the mathematics by applying the provided model.

<u>Suggested solution:</u>

The height of the building

$= 15.2 \text{ cm } \times 16 \times 17$

$= 4134.4 \text{ cm}$

$= 41 \text{ m}$

3. The teacher should emphasise that such a calculation is possible because we have made some assumptions. Besides, there are limitations in the model equation. The discussion in Question 3 thus enhances students' abilities in making assumptions and identifying limitations in modelling. The following are some possible discussion outcomes.
   - Assumptions:
     1. Uniform step heights (h)
     2. Perpendicular staircase

3. Same number of steps (n) for each storey (k)

- Limitations:
  The model overlooks the sections extending beyond the staircase area, such as the rooftop or other architectural components.

4. In reality, our planned approach is not always feasible due to situational constraints, such as inaccessible areas and safety concern. Through the discussion in Question 4, the teacher can enhance students' rule-abidingness and safety awareness during STEAM activities. The following are some possible discussion outcomes.
   1. Inaccessible areas: In some cases, certain sections of the building may be private areas and inaccessible for step counting and measurement.
   2. Safety concern: There may be a potential risk associated with entering building staircases, especially in unfamiliar or unsecured environments.

Activity 2A (refer to Worksheet 2)

This activity uses trigonometric ratios in height estimation, employing digital protractor to measure the angle of elevation. The integration of technology highlights its vital role in solving real-world problems and mathematical modelling.

Pedagogical recommendations:

1. The teacher can recall students' prerequisite knowledge of using trigonometric ratios in solving problems related to plane figures. Question 1 has provided the necessary information and assumption.

   Suggested solution:

   $$\tan 37.2° = \frac{BC}{5.63}$$
   $$BC = 5.63 \tan 37.2°$$
   $$= 4.27 \text{ m}$$

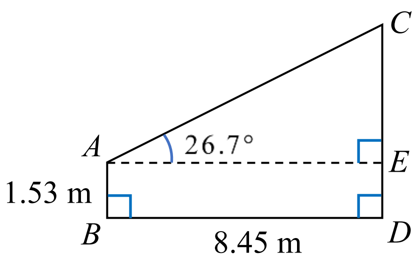
2. Similar to Question 1, Question 2 aims to recall students' prerequisite knowledge. Nevertheless, this question is more advanced because it takes student's eye level into account. Constructing a straight line perpendicular to the tree is required.

   Suggested solution:

   Draw $AE \perp CD$

   $AE = 8.45$ m

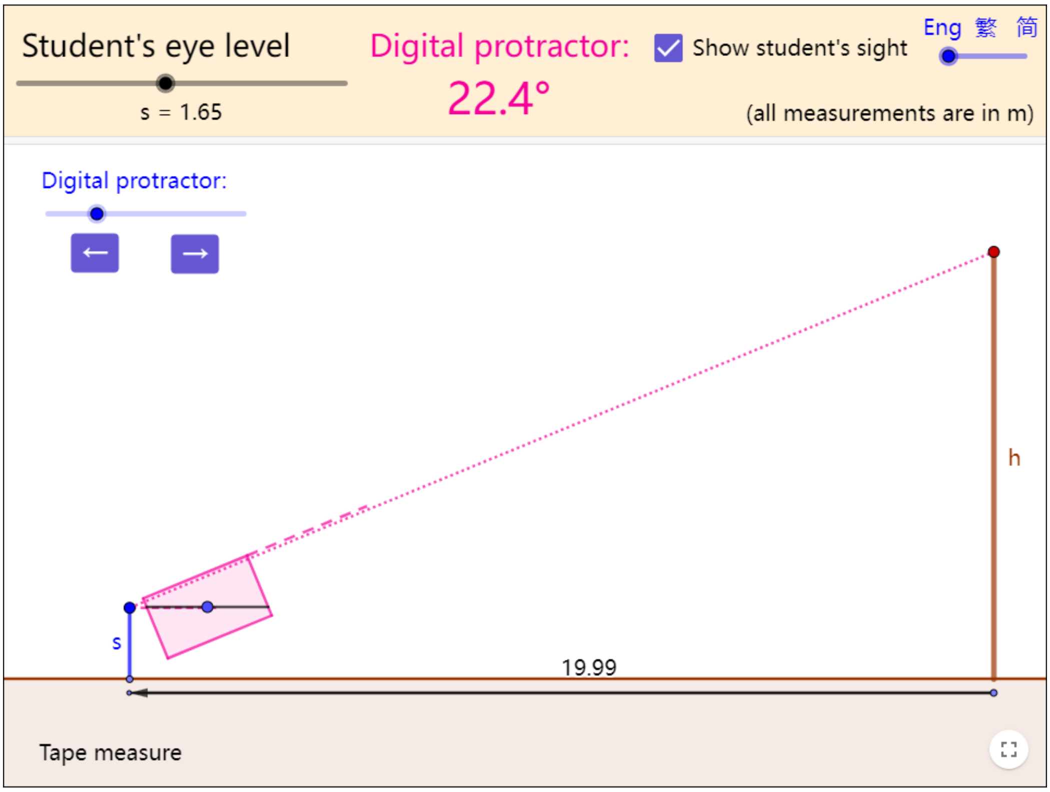
   $$\tan 26.7° = \frac{CE}{8.45}$$
   $$CE = 8.45 \tan 26.7°$$

   $$CD = CE + ED$$
   $$= 8.45 \tan 26.7° + 1.53$$
   $$= 5.78 \text{ m}$$

3. With the experience gained from Question 2, students are tasked to (a) make necessary assumptions and then (b) estimate the height of a building. Tape measures are required to measure a student's eye level and horizontal distance between the student and the building, and digital protractors are required to measure the angles of evaluation.

   The applet (https://www.geogebra.org/m/ymen6puf) provides a virtual simulation of measurement, making this activity feasible within classroom settings. The teacher can also use the applet to demonstrate the measurement process.



One set of possible measured values is as follows.

- Angle of elevation = 22.4°
- Student's eye level s = 1.65 m
- Horizontal distance between the student and the building = 19.99 m

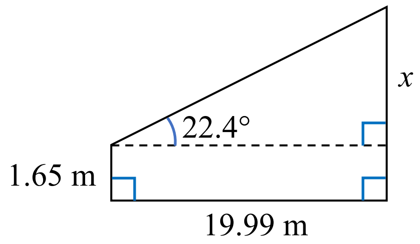
<u>Suggested solution:</u>

(a) Assumptions:

1. Vertical building: The angle between the ground and the building is exactly 90°.
2. Ground conditions: The ground is free of irregularities, as they can affect the accuracy when measuring the horizontal distance between the student and building.
3. Ground level: The student's standing position and the base of the building are at the same horizontal level.

(b) *[Based on the above set of possible measured values]*

From the student's eye, draw a horizontal line perpendicular to the building. Let x be the unknown, as shown in the figure.



$$\tan 22.4° = \frac{x}{19.99}$$

$$CE = 19.99 \tan 22.4°$$

$$\begin{aligned} h &= x + 1.65 \\ &= 19.99 \tan 22.4° + 1.65 \\ &= 9.89 \text{ m} \end{aligned}$$

$\therefore$ The height of the building is 9.89 m.

4. Toward the end of Activity 2A, the teacher can facilitate the discussion on the situational constraints. The following are some possible discussion outcomes.
   1. Weather conditions: Bad weather conditions (e.g., heavy fog or rain) can obscure the view and affect the accuracy of the angle measurement.
   2. Obstructions: There may be obstacles which block the way of measuring the distance between the student's standing position and the base of the building.

Activity 2B (refer to Worksheet 2)

This enrichment activity expands the use of trigonometric ratios in height estimation, particularly in situations where measuring horizontal distances is not entirely feasible.

Pedagogical recommendations:

5. The teacher can introduce students to the approach using Example 5, where measuring the length of *CD* is not feasible.
   - Step 1: Measure the distance between two points (i.e., *A* and *D*) on the ground. Notice that the two points and the base of the building should be colinear (i.e., *A*, *D* and *C* are colinear).
   - Step 2: Measure the angle of elevation from each of these points to the top of the building (i.e., $\angle BAC$ and $\angle BDC$).
   - Step 3: Solve the mathematics with the knowledge of solving simultaneous linear equations in two unknowns.

6. Students apply what they have learnt to solve Quick Practice 6. The teacher can thus check their understanding of the approach and provide feedback accordingly.

   Suggested solution:

   Let $BC = h$ m and $DC = x$ m.

   In ΔBCD,

   $$\tan 60° = \frac{h}{x}$$

   $$x = \frac{h}{\tan 60°} \quad \cdots\cdots(1)$$

   In ΔABC,

   $$\tan 30° = \frac{h}{x+9}$$

   $$x+9 = \frac{h}{\tan 30°} \quad \cdots\cdots(2)$$

   By substituting (1) into (2), we have

   $$\frac{h}{\tan 60°} + 9 = \frac{h}{\tan 30°}$$

   $$9 = \frac{h}{\tan 30°} - \frac{h}{\tan 60°}$$

   $$9 = h(\frac{1}{\tan 30°} - \frac{1}{\tan 60°})$$

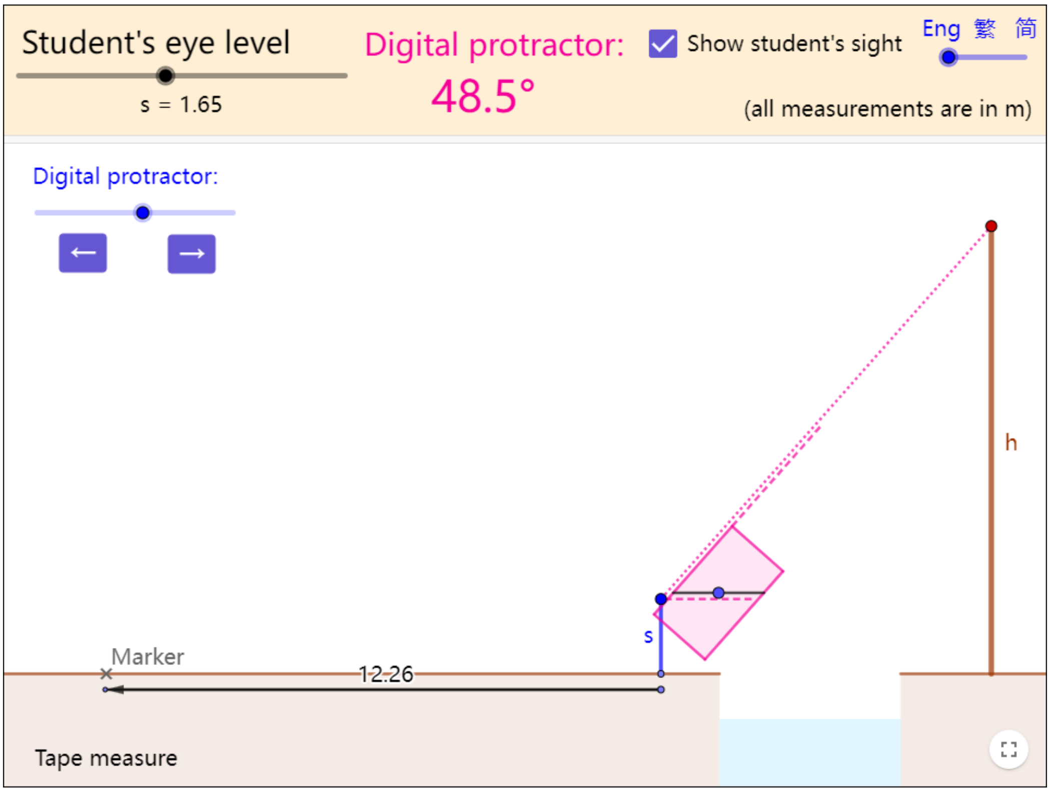
   $$9 \div (\frac{1}{\tan 30°} - \frac{1}{\tan 60°}) = h$$

   $$h = 7.79$$

   $\therefore$ $BC = 7.79$ m

7. With the experience gained from Quick Practice 6, students are tasked to (a) make necessary assumptions and then (b) estimate the height of the building. Tape measures are required to measure a student's eye level and horizontal distance between the first and second positions, and digital protractors are required to measure the angles of evaluation.

The applet (https://www.geogebra.org/m/djurma4f) provides a virtual simulation of measurement, making this activity feasible within classroom settings. The teacher can also use the applet to demonstrate the measurement process.



One set of possible measured values is as follows.

- The first angle of elevation = 22.8°
- The second angle of elevation = 48.5°
- Student's eye level s = 1.65 m
- Horizontal distance between the first and second positions = 12.26 m

Suggested solution:

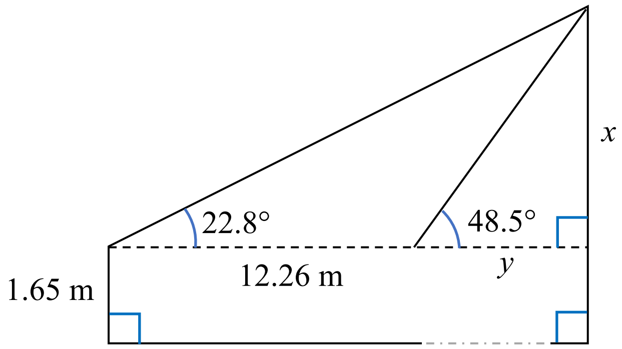
(a) Assumptions:

Ground level: The student's two standing positions and the base of the building are at the same horizontal level and colinear.

(b) *[Based on the above set of possible measured values]*

From the student's eye, draw a horizontal line perpendicular to the building.

Let x and y be the unknowns, as shown in the figure.



$$\tan 48.5° = \frac{x}{y}$$

$$y = \frac{x}{\tan 48.5°} \quad \cdots\cdots(1)$$

$$\tan 22.8° = \frac{x}{y+12.26}$$

$$y+12.26 = \frac{x}{\tan 22.8°} \quad \cdots\cdots(2)$$

By substituting (1) into (2), we have

$$\frac{x}{\tan 48.5°} + 12.26 = \frac{x}{\tan 22.8°}$$

$$12.26 = \frac{x}{\tan 22.8°} - \frac{x}{\tan 48.5°}$$

$$12.26 = x\left(\frac{1}{\tan 22.8°} - \frac{1}{\tan 48.5°}\right)$$

$$12.26 \div \left(\frac{1}{\tan 22.8°} - \frac{1}{\tan 48.5°}\right) = x$$

$$x = 8.205165382$$

$$h = 8.205165382 + 1.65$$

$$= 9.86$$

$\therefore$ The height of the building is 9.86 m.

<u>Activity 3 (refer to Worksheet 3)</u>

In this activity, students will create a virtual 3D model of a building using GeoGebra. The teacher can deliver the activity in class or assign it as a post-class task.

Pedagogical recommendations:

1. Students are tasked to use online maps to find the lateral dimension of the building. The teacher can suggest them to screen-capture and download the map of the target building alongside the scale. With this scale, it will be easier to get the proportions right when creating the 3D model.

2. https://www.geogebra.org/classic is the online application of GeoGebra. Alternatively, the teacher and students can install GeoGebra on computers. Please visit: https://www.geogebra.org/download

3. The teacher can introduce students to the IT skills of using GeoGebra in creating the 3D model. The teacher can use the step-by-step instructions in Worksheet 3 in which a 96-metre-tall building is used as an example.

4. Toward the end of this activity, the teacher can facilitate the discussion on the assumptions of this 3D model. The following are some possible discussion outcomes.
    1. Building shape: The building is a prism, having a uniform cross-section along its entire height.
    2. The top of the building: The roof or uppermost floor of the building is perfectly level.

Concluding remarks:

Overall, students engaged with two distinct approaches to height estimation and then created their 3D virtual model. They also explored the assumptions, limitations, and situational constraints inherent in each approach.

To conclude this activity, the teacher can summarise and compare the two approaches:

- Activity 1: Estimating building height through measuring the height of individual step and total counts is straightforward but may face situational constraints, such as inaccessible areas.
- Activity 2: Estimating building height through an angle of elevation and horizontal distance does not require entering the building but requires tools (e.g., digital protractor) for angle measurement.

There may be some other approaches. For example, using shadow lengths and similar triangles allows estimating building height without entering the building or requiring a digital protractor. However, we may encounter situational constraints related to weather conditions (e.g., rainy or cloudy days with insufficient sunlight will make measuring the lengths of shadows impossible). Therefore, the choice of an appropriate approach to estimation and modelling should be based on our specific situation and the available resources.

References:

Galbraith, P., & Holton, D. (2018). *Mathematical modelling: A guidebook for teachers and teams.* Australia: Australian Council for Educational Research.

Yong, D., Levy, R., & Lape, N. (2015). Why no difference? A controlled flipped classroom study for an introductory differential equations course. *PRIMUS, 25*(9–10), 907–921.

Suggested lesson plans and teaching flow

Teaching time: 70 minutes or a double lesson

Time (mins)	Objectives	Teaching activities and processes	Resources/ remarks
20	• To arouse students' interest	1. The teacher arouses students' interest by discussing the real-world scenario.	WS cover page
	• To establish the context of height estimation	2. The teacher arouses students' interest by discussing the ways of estimating the length and the height of a podium. 3. The teacher points out the difference between Q1(a) and Q1(b), thereby highlighting the necessity of making assumptions in solving real-world problems.	WS1 Q1
	• To enhance abilities in identifying assumptions made	4. Students discuss in groups the main assumption made and determine whether it is reasonable.	
	• To apply modelling outcomes	5. Using the approach presented in Question 2, the teacher introduces students to the mathematical modelling framework. 6. Students apply the provided model to solve the problem.	WS1 Q2
	• To enhance abilities in identifying assumptions, limitations, and situational constraints in modelling	7. Students discuss in groups the assumptions and limitations of the model. 8. Students discuss in groups the possible situational constraints of using the modelling approach.	WS1 Q3–4

Time (mins)	Objectives	Teaching activities and processes	Resources/ remarks
20	• To recall prerequisite knowledge • To estimate heights using trigonometric ratios • To enhance abilities in making assumptions • To enhance abilities in identifying situational constraints in modelling	1. The teacher uses examples as warm-up exercises to recall students' knowledge of using trigonometric ratios in solving problems related to plane figures. 2. The teacher may use the applet to demonstrate the measurement process. 3. Students discuss in groups the necessary assumptions in the estimation. 4. Students measure a student's eye level and the horizontal distance between him/her and the building using a tape measure. They also measure the angle of evaluation of the top of the building from his/her eye using a digital protractor. 5. Students estimate the height of the building based on their measured values. 6. Students discuss in groups the possible situational constraints in the estimation.	WS2A Q1–2 WS2A Q3 WS2A Q4
[20]*	• To estimate heights using trigonometric ratios (advanced)	1. The teacher uses examples and quick practices to introduce how to use trigonometric ratios in height estimation when measuring horizontal distances is not entirely feasible.	WS2B Q5–6

Time (mins)	Objectives	Teaching activities and processes	Resources/ remarks
		2. The teacher may use the applet to demonstrate the measurement process. 3. Students discuss in groups the necessary assumptions in the estimation. 4. Students measure a student's eye level and the horizontal distance between his/her first and second positions using a tape measure. They also measure the first and second angles of evaluation of the building from his/her eye using a digital protractor. 5. Students estimate the height of the building based on their measured values.	WS2B Q7
25	• To create a 3D virtual model	1. Students use GeoGebra to create a virtual 3D model of the building. 2. Students discuss in groups the assumptions involved in the 3D model.	WS3 Q1–3 WS3 Q4
5	• To conclude the activity	1. The teacher summarises and compares the two approaches to height estimation. 2. The teacher emphasises the strengths and limitations of each approach.	

* Based on student abilities and school contexts, teachers can conduct either Activity 2A or Activity 2B.